Chceme odvodiť vzťah: $$c_{m+1}(n) = \sum_k h_{mr}(k-2n)c_m(k)$$

Vieme, že platí: $$\varphi_{m,n}(t) = 2^{-m/2}\varphi\left(2^{-m}t - n\right)$$

$$\varphi(t) = \sqrt{2}\sum_{n=-\infty}^{\infty} h_{mr}(n)\varphi(2t-n)$$

Vynásobme obe strany druhej rovnice $2^{-m/2}$ a za t dosaďme $2^{-m}t - n$:

$$2^{-m/2}\varphi\left(2^{-m}t - n\right) = 2^{-m/2}.2^{1/2}\sum_k h_{mr}(k)\varphi\left(2(2^{-m}t-n)-k\right)$$

$$\varphi_{m,n}(t) = \sum_k h_{mr}(k)2^{-(m-1)/2}\varphi\left(2^{-(m-1)}t - 2n - k\right) =$$

$$= \sum_k h_{mr}(k)\varphi_{m-1,2n+k}$$

$$c_m(n) = \left\langle f(t), \varphi_{m,n}(t)\right\rangle = \left\langle f(t), \sum_k h_{mr}(k)\varphi_{m-1,2n+k}\right\rangle = \sum_k h_{mr}(k)\left\langle f(t), \varphi_{m-1,2n+k}\right\rangle =$$

$$= \sum_k h_{mr}(k)\, c_{m-1}(2n+k) = \sum_k h_{mr}(k-2n)\, c_{m-1}(k)$$

T.j: $$c_{m+1}(n) = \sum_k h_{mr}(k-2n)c_m(k)$$

DWT v maticovom tvare.

Uvažujme nasledovné tvary rovníc:

Rozklad(analýza): $c_{m+1}(n)=\sum_k \tilde{h}_{mr}(k-2n)c_m(k)$ $\quad$ $d_{m+1}(n)=\sum_k \tilde{g}_{mr}(k-2n)c_m(k)$

Rekonštrukcia(syntéza): $c_m(n)=\sum_k h_{mr}(n-2k)c_{m+1}(k)+\sum_k g_{mr}(n-2k)d_{m+1}(k)$

Tieto vzťahy môžeme prepísať do maticového tvaru ako transformácie:

$$\begin{pmatrix}\mathbf{C}_{m+1}\\ \mathbf{D}_{m+1}\end{pmatrix}=\mathbf{T}_a\mathbf{C}_m \quad \mathbf{C}_m=\mathbf{T}_s\begin{pmatrix}\mathbf{C}_{m+1}\\ \mathbf{D}_{m+1}\end{pmatrix}$$

, kde T_a, T_s sú ***štvorcové*** transformačné matice pre analýzu, resp. pre syntézu a $\mathbf{C}_m$ resp. $\mathbf{D}_m$ sú stĺpcové vektory (matice) :

$$\mathbf{C}_m=\left(c_m(0),c_m(1),\ldots,c_m(N_m-1)\right)^T$$
$$\mathbf{D}_m=\left(d_m(0),d_m(1),\ldots,d_m(N_m-1)\right)^T$$

kde veľkosť vektorov je:

$$N_m=2^{-m}N_0$$

Pozn.: *v ďalšom texte budeme* $\tilde{h}_{mr}$, h_{mr} , g_{mr} , $\tilde{g}_{mr}$ *používať bez označenia "mr".*

Maticový zápis pri periodickom rozšírení signálu

$$\widetilde{\mathbf{H}}_m = \overbrace{\begin{pmatrix} \tilde{h}(0) & \tilde{h}(1) & \dots & \dots & \dots & \tilde{h}(-1) \\ \dots & \tilde{h}(-1) & \tilde{h}(0) & \tilde{h}(1) & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & \tilde{h}(-1) & \tilde{h}(0) & \tilde{h}(1) \end{pmatrix}}^{N_m} \qquad \widetilde{\mathbf{G}}_m = \overbrace{\begin{pmatrix} \tilde{g}(0) & \tilde{g}(1) & \dots & \dots & \dots & \tilde{g}(-1) \\ \dots & \tilde{g}(-1) & \tilde{g}(0) & \tilde{g}(1) & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & \tilde{g}(-1) & \tilde{g}(0) & \tilde{g}(1) \end{pmatrix}}^{N_m} \left.\vphantom{\begin{pmatrix} a\\a\\a\\a \end{pmatrix}}\right\} \frac{N_m}{2}$$

$$\mathbf{H}_m = N_m \left\{ \begin{pmatrix} h(0) & \vdots & \dots & \vdots \\ h(1) & h(-1) & \dots & \vdots \\ \vdots & h(0) & \dots & \vdots \\ \vdots & h(1) & \dots & h(-1) \\ \vdots & \vdots & \dots & h(0) \\ h(-1) & \vdots & \dots & h(1) \end{pmatrix} \right. \qquad \mathbf{G}_m = N_m \left\{ \begin{pmatrix} g(0) & \vdots & \dots & \vdots \\ g(1) & g(-1) & \dots & \vdots \\ \vdots & g(0) & \dots & \vdots \\ \vdots & g(1) & \dots & g(-1) \\ \vdots & \vdots & \dots & g(0) \\ g(-1) & \vdots & \dots & g(1) \end{pmatrix} \right.$$

Analýza: $\mathbf{C}_{m+1} = \widetilde{\mathbf{H}}_m \mathbf{C}_m \qquad \mathbf{D}_{m+1} = \widetilde{\mathbf{G}}_m \mathbf{C}_m$

$$\begin{pmatrix} \mathbf{C}_{m+1} \\ \mathbf{D}_{m+1} \end{pmatrix} = \begin{pmatrix} \widetilde{\mathbf{H}}_m \\ \widetilde{\mathbf{G}}_m \end{pmatrix} \mathbf{C}_m \qquad \rightarrow \qquad \mathbf{T}_a = \begin{pmatrix} \widetilde{\mathbf{H}}_m \\ \widetilde{\mathbf{G}}_m \end{pmatrix}$$

Syntéza: $\mathbf{C}_m = \begin{pmatrix} \mathbf{H}_m & \mathbf{G}_m \end{pmatrix} \begin{pmatrix} \mathbf{C}_{m+1} \\ \mathbf{D}_{m+1} \end{pmatrix} \qquad \rightarrow \qquad \mathbf{T}_s = \begin{pmatrix} \mathbf{H}_m & \mathbf{G}_m \end{pmatrix}$

V prípade Haarovej DWT:

- **Rozklad (= dopredná transformácia)**

$$c_{m+1}(n) = \sum_k \tilde{h}_{mr}(k-2n)c_m(k) \qquad d_{m+1}(n) = \sum_k \tilde{g}_{mr}(k-2n)c_m(k)$$

$$\mathbf{C}_{m+1} = \tilde{\mathbf{H}}_m \mathbf{C}_m \qquad \mathbf{D}_{m+1} = \tilde{\mathbf{G}}_m \mathbf{C}_m$$

$$\tilde{h}_{mr} = \frac{\sqrt{2}}{2}(1,1) \qquad \tilde{g}_{mr} = \frac{\sqrt{2}}{2}(1,1)$$

$$\tilde{\mathbf{H}}_m = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \qquad \tilde{\mathbf{G}}_m = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

Príklad vstupného signálu:

$$c_0(n) = (1,2,3,4,5,6,7,8)^T \qquad N_0 = 8$$

$$c_1(n) = \frac{\sqrt{2}}{2}(3,7,11,15)^T \qquad d_1(n) = \frac{\sqrt{2}}{2}(-1,-1,-1,-1)^T$$

$$\mathbf{T}_a = \begin{pmatrix} \tilde{\mathbf{H}}_m \\ \tilde{\mathbf{G}}_m \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

Reprezentácia signálu po jednom transformačnom kroku:

$$\left(c_1(n), d_1(n)\right) = \frac{\sqrt{2}}{2}\left(3,7,11,15,-1,-1,-1,-1\right)^T$$

➔ Ďalšie kroky analogicky ... (ešte 2 kroky)

- **Rekonštrukcia (= spätná transformácia)**

$$c_m(n) = \sum_k h_{mr}(n-2k)c_{m+1}(k) + \sum_k g_{mr}(n-2k)d_{m+1}(k)$$

$$\mathbf{C}_m = \begin{pmatrix} \mathbf{H}_m & \mathbf{G}_m \end{pmatrix} \begin{pmatrix} \mathbf{C}_{m+1} \\ \mathbf{D}_{m+1} \end{pmatrix} = \mathbf{T}_s \begin{pmatrix} \mathbf{C}_{m+1} \\ \mathbf{D}_{m+1} \end{pmatrix}$$

$$\mathbf{H}_m = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{G}_m = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \mathbf{T}_s = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Vektory bázy, do ktorej sme transformovali, sú stĺpcové vektory z matice $\mathbf{T}_s$.

Všeobecne (opakovanie)

$$\mathbf{T_a T_s} = \mathbf{I}_N = \mathbf{T_s T_a}$$

Vyjadrime:

$$\mathbf{I}_N = \begin{pmatrix} \mathbf{I_{N/2}} & \mathbf{0_{N/2}} \\ \mathbf{0_{N/2}} & \mathbf{I_{N/2}} \end{pmatrix} = \mathbf{T}_a \mathbf{T}_s = \begin{pmatrix} \widetilde{\mathbf{H}} \\ \widetilde{\mathbf{G}} \end{pmatrix} \begin{pmatrix} \mathbf{H} & \mathbf{G} \end{pmatrix} = \begin{pmatrix} \widetilde{\mathbf{H}}\mathbf{H} \\ \widetilde{\mathbf{G}} \end{pmatrix} \begin{pmatrix} \mathbf{H} & \mathbf{G} \end{pmatrix} = \begin{pmatrix} \widetilde{\mathbf{H}}\mathbf{H} & \widetilde{\mathbf{H}}\mathbf{G} \\ \widetilde{\mathbf{G}}\mathbf{H} & \widetilde{\mathbf{G}}\mathbf{G} \end{pmatrix}$$

z čoho vyplýva:

$$\widetilde{\mathbf{H}}\mathbf{H} = \widetilde{\mathbf{G}}\mathbf{G} = \mathbf{I} \qquad \widetilde{\mathbf{H}}\mathbf{G} = \widetilde{\mathbf{G}}\mathbf{H} = 0$$

T.j. impulzové charakteristiky filtrov sú ortogonálne k párnym posunom svojich duálov a ortogonálne navzájom „nakríž“ = podmienky ***biortogonality***.

Pre Ortogonálne DWT platí nasledovná podmienka :

$$\tilde{h}_{mr}(n) = h_{mr}(n)\ ,\ \tilde{g}_{mr}(n) = g_{mr}(n) \qquad g_{mr}(n) = \pm(-1)^n h_{mr}(M-n)$$